Autour des séries de Fourier et EDPs nonlinéaires

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Outline

- Lecture 1 : Introduction
  - History
  - Two physical problems : Heat equation & Vibrating string
  - Convergence of Fourier series
  - Some elementary properties of Fourier series & Fourier transform
  - What about other PDEs ?
- Lecture 2 : Nonlinear Schrödinger equation I
  - Function spaces using Fourier series
  - Cauchy problem
- Lecture 3 : Nonlinear Schrödinger equation II
  - Motivation from the wave turbulence theory
  - Long time behavior (possible growth of solutions)

History

Joseph Fourier (1768–1830).

- Fourier series
- Fourier transform
- Discrete Fourier transform
- Fast Fourier transform
- ...

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

The Fourier series is named in honor of *Joseph Fourier*(1768–1830), who made important contributions to the study of trigonometric series.

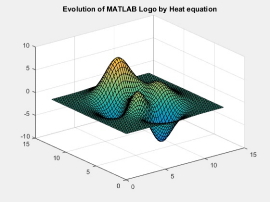
Fourier introduced the series for the purpose of solving the **heat equation** in a metal plate, publishing his initial results *Mémoire sur la propagation de la chaleur dans les corps solides* in 1807.

History

The heat equation is a partial differential equation

$$\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t)$$

Question : How to solve it ?



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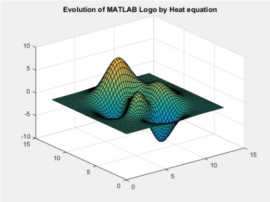

Heat equation

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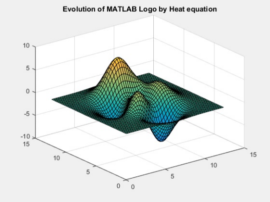

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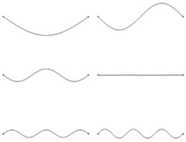
The theory of the heat equation was first developed by Fourier in **1822**.

His idea was to model a complicated heat source as a linear combination of simple sine/cosine waves, and to write the solution as a linear combination of the corresponding eigensolutions. This linear combination is called the Fourier series.

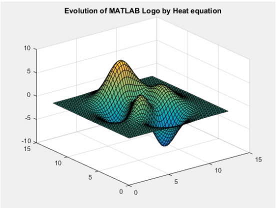
Two physical problems

- I. String vibration



String vibration

- II. Heat equation



Heat equation

Convergence of Fourier series

When we use the method of separation of variables to solve PDE, we represent a periodic function f by a trigonometric series of the form

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$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}$$

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Euler (1707-1783)

If $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is an absolutely integrable function, its Fourier coefficients $\hat{f} : \mathbb{Z} \to \mathbb{C}$ are defined by the formula

$$\hat{f}(k) := \int_{\mathbb{R}/\mathbb{Z}} f(x) e^{-2\pi ikx} dx.$$

Fourier (1768–1830)

The trigonometric series with these coefficients, $\sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi ikx}$ is called the Fourier series of f.

Convergence of Fourier series

Question : When and in what sense the series $\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi ikx}$ represents the function f ?

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Consider the partial summation operators

$$S_N f(x) := \sum_{|k|<N} \hat{f}(k)e^{2\pi ikx}.$$

Dirichlet wrote the partial sums as follows :

$$(S_N f)(x) = \sum_{|k|<N} e^{2\pi ikx} \int_0^1 f(y)e^{-2\pi iky}dy$$
$$= \int_0^1 f(y) \sum_{|k|<N} e^{2\pi ik(x-y)}dy = (D_N * f)(x)$$

Dirichlet (1805 - 1859)

where D_N is the Dirichlet kernel

$$D_N(y) = \sum_{|k|<N} e^{2\pi iky} = \frac{\sin(\pi(2N+1)y)}{\sin(\pi y)}.$$

Convergence of Fourier series

Two criteria for pointwise convergence.

1. Dini's Criteria
If for some x there exists $\delta > 0$ such that

$$\int_{|t|<\delta} \Big|\frac{f(x+t)-f(x)}{t}\Big| dt < \infty,$$

then

$$\lim_{N\to\infty} S_N f(x) = f(x).$$

Dini (1845 - 1918)

2. Jordan's Criteria
If f is a function of bounded variation in a neighborhood of x, then

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}\Big(f(x+) + f(x-)\Big)$$

Jordan (1838 - 1922)

Theorem (Riemann Localization Principle)

If f is zero in a neighborhood of x, then

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Riemann (1826 -1866)

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If $f \in L^1(\mathbb{T})$ then

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P. du Bois-Reymond proved that there exists a continuous function whose Fourier series diverges at a point.

Riemann (1826 -1866)

Lebesgue (1875-1941)

P. du Bois-Reymond (1831-1889)

Convergence of Fourier series

Two types of convergence :

- Question 1 : Does $\lim_{N\to\infty} \|S_N f - f\|_p = 0$ for $f \in L^p(\mathbb{T})$?
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Theorem (Almost everywhere convergence)

- *(Kolmogorov, 1923). There exists $f \in L^1(\mathbf{R}/\mathbf{Z})$ such that $S_N f(x)$ is unbounded in N for almost every x.*
- *(Carleson, 1966, conjectured by Lusin, 1913) For every $f \in L^2(\mathbf{R}/\mathbf{Z})$, $S_N f(x)$ converges to $f(x)$ as $N \to \infty$ for almost every x.*
- *(Hunt, 1967). For every $1 < p \le \infty$ and $f \in L^p(\mathbf{R}/\mathbf{Z})$, $S_N f(x)$ converges to $f(x)$ as $N \to \infty$ for almost every x.*

Kolmogorov (1903-1987)

Carleson (1928-)

Some elementary properties

Given a function $f \in L^1(\mathbb{R})$, the following is a list of properties of the Fourier transform:

- linearity : $(\widehat{\alpha f + \beta g}) = \alpha \hat{f} + \beta \hat{g}$
- Riemann-Lebesgue : $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$
- $\|\hat{f}\|_\infty \leq \|f\|_{L^1}$ and f is continuous.
- $\widehat{f * g} = \hat{f}\hat{g}$
- $\widehat{\partial_x f}(\xi) = 2\pi i \xi \hat{f}(\xi)$

Reference :

- *Fourier analysis* by Javier Duoandikoetxea.
- *Fourier analysis : an introduction* by Elias M. Stein & Rami Shakarchi.

Other PDE

Nonlinear Schrödinger equation (NLSE)

$$i\partial_t u + \Delta u = |u|^2 u, \qquad u(x,t) \in \mathbb{C}, x \in \mathbb{T}^2.$$

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Question :

- Cauchy theory?
- Long time behavior?

À demain !